On the Investment-Uncertainty Relationship: A Game Theoretic Real Option Approach

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Abstract:

This paper examines the effect of uncertainty on investment timing in a game theoretical real option model. We extend the settings of Gryglewicz et al. (2008), Wong (2007), and Sarkar (2000) by a more general assumption, i.e. the investment is also influenced by the actions of a second player. The results show that a U-shaped investment-uncertainty relationship generally sustains. However, timing of an investment occurs inefficiently late. Moreover, we show that the influence of uncertainty on the associated first-mover advantage becomes ambiguous, too.

Keywords: real option, investment, uncertainty, managerial flexibility, game theory

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1. Introduction

The timing of strategic investments is among the most common problems in corporate finance and corporate strategy, respectively. Due to inherent uncertainty an investor has to face the trade-off between early commitment to profit from first-mover advantages and late commitment due to maintenance of flexibility. The latter case is a result of the irreversible nature of most strategic investments, i.e. once an investment is made the incurred sunk costs cannot be recovered should the project be abandoned at a later stage. Option-based valuation of investments has been proposed as an analytical tool to address these issues and the literature has provided various examples that give guidance on how to optimally time an investment under uncertainty (see e.g., Dixit and Pindyck, 1994; Trigeorgis, 1998; Kort et al., 2010).

Real options express the flexibility assigned to a decision, i.e. for example the decision to delay an investment or to abandon an investment project without being obliged to.[1] Here, the simple investment opportunity represents a perpetual American Call option and the resulting investment-uncertainty relationship is of negative sign, i.e. the higher the uncertainty the higher the propensity to postpone the investment. Only recently, however, Sarkar (2000), Lund (2005), Wong (2007), and Gryglewicz et al. (2008), have shown that this relationship is not necessary monotonic. In particular, the authors correct the usual assumption in the extent literature that the risk-adjusted return on a project is invariant to the volatility of an investment's returns. As a result, situations may occur where increased uncertainty increases the propensity to invest and promotes early

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[1] For example Dixit and Pindyck (1994) .

exercise of a real option. The question that arises is whether these results hold once two ore more individuals decide on the timing of an investment.

This paper extends the setting of Gryglewicz et al. (2008) by a more general assumption, i.e. the investment is also influenced by the actions of a second player. Such a setting may exist in any situations where two individuals jointly decide on the implementation of investment projects, such as in manager-shareholder or customer-supplier relationships. We show that a U-shaped investment-uncertainty relationship generally sustains. However, timing of an investment occurs inefficiently late. Specifically, the socially inefficiency is the more pronounced the higher the fraction of costs the second player has to bear. While these effects are partly due to a first-mover advantage we furthermore show that the resulting first-mover advantage is also non-monotonically affected by uncertainty. We thereby add also new insights to another stream of recent real options literature that addresses the outcome of bargaining under uncertainty (See e.g., Morellec and Zhdanov, 2005; Hackbarth and Morellec, 2008; Lambrecht, 2004; Lee, 2004; Cvitanić et al., 2011; Lukas and Welling, 2012).

The rest of the paper is organized as follows. Section two presents the model and characterizes the optimal investment threshold and surplus distribution. Section three illustrates numerically the impact of uncertainty on timing and on the size of the first-mover advantage. Finally, section four concludes and lays out several directions for future research.

2. The Model

In the following, we deviate from a canonical real options model à la Dixit and Pindyck (1994) . In particular, consider an investment project whose net cash flow per unit time, $x(t)$, can be expressed by the following stochastic differential equation:

$$dx(t) = \alpha x(t)dt + \sigma x(t)dW(t), \qquad x(0) = x_0, \tag{1}$$

with $\sigma \in \mathbb{R}_+$ as the volatility of the cash flow stream, where $\lambda = (\mu_M - r)/\sigma_M$ denotes the market price of risk, μ_M and σ_M the return and volatility of the market portfolio and r the riskless rate of interest. The investment project's life is finite of length T. Hence, the project's present value $V(t)$ is the given by:[2]

$$V(t) = \mathbb{E}[\int_0^T x(s)e^{-\mu(s-t)}ds] = x_0 \int_0^T e^{-(\mu-\alpha)}ds = x_0 \frac{1-e^{-(r+\lambda\rho\sigma-\mu)T}}{r+\lambda\rho\sigma-\mu}.$$

Consequently, the dynamics of $V(t)$ are also governed by a geometric Brownian motion, i.e.:

$$dV(t) = \alpha V(t)dt + \sigma V(t)dW(t) \quad V(0) = V_0. \tag{2}$$

In contrast to the previous mentioned literature, however, we will assume that investment in the project requires the actions of two participating individuals A and B. Individual A compensates B by transferring a portion of the asset value $\psi V(x)$ to the second individual B upon investment while B times the initiation of

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[2] In a non-game theoretic setting, the assumption of a finite project lifetime guarantees that the U-shaped pattern persists once cash flow dynamics are directly modelled, see e.g. Gryglewicz et al. (2008).

the investment project. Moreover, investment in the project is associated with sunk costs I and we will assume that A and B incur the fraction $(1 - \varepsilon)I$ and εI, respectively. Here, $\varepsilon \in [0,1]$ expresses the distribution of sunk costs between the two individuals and is provided exogenously. The simplest example for such an investment setting would be the initiation of a greenfield joint venture where profits and costs are generally shared or a principal-agent setting under complete information.

We assume that time is continuous, i.e. $t \in [t_0, \infty)$ and rely on a Markovian Perfect Nash Equilibrium to determine the equilibrium strategy for both parties. In particular, as in Betton and Morán (2003), A optimally defines ψ in stage one and conditional on the offered premium ψ B will choose a threshold value $V^*(\psi)$ or $x^*(\psi) := \frac{1 - e^{-(r + \lambda \rho \sigma - \mu)T}}{r + \lambda \rho \sigma - \mu} V^*(\psi)$, respectively, in stage two at which the offer will be accepted. Thus $t^* := inf\{t \geq t_0 | x(t) \geq x^*\} = inf\{t \geq t_0 | V(t) \geq V^*\}$ is the time of investing. However, B has not to decide immediately at time t_0 of the offer whether it accepts or rejects the offer. Rather, we assume that B can postpone the decision. To be more precise, while A has the action set $\psi \in (0, \infty)$, B has at every point in time the action set {accept, wait}. This degree of managerial flexibility B possesses can be interpreted as a real option. Exercising the option right refers to accepting the offer by initiating the investment project.[3]

Consequently, the value of the option to invest in the project held by the reacting party B is the solution of the following maximization problem in stage two:

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[3] We will assume that this managerial flexibility is not limited by a fixed maturity date. Therefore the possibility to accept the offer is a perpetual real option.

$$F(x) = \max_{\tau} \mathbf{E}\left[\left(\frac{(1 - e^{-\delta T})}{\delta}\psi x_{\tau} - \varepsilon I\right)e^{-(r+\lambda\rho\sigma)\tau}\right], \tag{3}$$

where **E[…]** denotes the expectations operator and $\delta = r + \lambda\rho\sigma - \mu$ denotes the convenience yield (or rate of shortfall) of the investment opportunity. Hence, the value $F(x(t))$ of the investment option is solution to the differential equation

$$\frac{1}{2}\sigma^2 x(t)^2 F''(x(t)) + (\alpha - \lambda\rho\sigma)x(t)F'(x(t)) - rF(x(t)) = 0 \tag{4}$$

with the boundary conditions

$$F(0) = 0, \tag{5}$$

$$F(x^*(\psi)) = \frac{(1 - e^{-\delta T})}{\delta}\psi x^*(\psi) - \varepsilon I \tag{6}$$

and

$$F'(x^*(\psi)) = \frac{(1 - e^{-\delta T})}{\delta}\psi, \tag{7}$$

where equation (5)-(7) are the zero-boundary condition, the value-matching condition and the smooth-pasting condition, respectively. Solving equation (4) yields:

$$F(x^*(\psi)) = \begin{cases} \left(\dfrac{(1 - e^{-\delta T})}{\delta}\psi V x^*(\psi) - \varepsilon I\right)\left(\dfrac{x}{x^*(\psi)}\right)^{\beta} & \text{if } x < x^*(\psi) \\[2ex] \dfrac{(1 - e^{-\delta T})}{\delta}\psi x - \varepsilon I & \text{if } x \geq x^*(\psi) \end{cases} \tag{8}$$

with

$$\beta = \frac{1}{2} - \frac{\alpha - \lambda\rho\sigma}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\alpha - \lambda\rho\sigma}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1 \tag{9}$$

and

$$x^*(\psi) = \frac{\delta}{(1 - e^{-\delta T})} \frac{\beta}{\beta - 1} \frac{\varepsilon I}{\psi}. \tag{10}$$

In contrast, the bidding firm A will choose ψ in stage one such that it maximizes

$$f(\psi) = \max_{\psi} \mathbf{E}\left[\left(\frac{(1 - e^{-\delta T})}{\delta}(1 - \psi)x^*(\psi) - (1 - \varepsilon)I\right)e^{-(r + \lambda\rho\sigma)t^*}\right]$$

$$= \max_{\psi}\left(\left(\frac{(1 - e^{-\delta T})}{\delta}(1 - \psi)x^*(\psi) - (1 - \varepsilon)I\right)\left(\frac{x_0}{x^*(\psi)}\right)^\beta\right), \tag{11}$$

subject to the other party's reaction function, i.e. $x^*(\psi)$.

Solving $\partial\left(\left(\frac{(1 - e^{-\delta T})}{\delta}(1 - \psi)x^*(\psi) - (1 - \varepsilon)I\right)\left(\frac{x_0}{x^*(\psi)}\right)^\beta\right)/\partial\psi = 0$ we get the

following proposition.

Proposition 1: *The optimal offered portion ψ results to:*

$$\psi = \frac{(\beta - 1)\varepsilon}{(\beta - 1) + \varepsilon}. \tag{12}$$

Using this result in equation (10) leads to the following proposition.

Proposition 2: *The optimal timing threshold x^* is given by:*

$$x^* = \frac{\delta}{(1 - e^{-\delta T})}\left(1 + \frac{\varepsilon}{\beta - 1}\right)\frac{\beta}{\beta - 1}I. \tag{13}$$

In a cooperative framework, i.e. the parties act as a central planer, the optimal timing threshold is

$$x_{eff}^* = \frac{\delta}{(1 - e^{-\delta T})}\frac{\beta}{\beta - 1}I. \tag{14}$$

Hence, we have

$$x^* = \left(1 + \frac{\varepsilon}{\beta - 1}\right)x_{eff}^*. \tag{15}$$

Considering that the optimal timing threshold in Gryglewicz et al. (2008) equals the efficient timing threshold x^*_{eff} , i.e. equation (14), we come to the following intermediate result:

Proposition 3a: *Analogous to Gryglewicz et al. (2008) the influence of uncertainty on the optimal timing threshold* x**for finite-lived investment projects is non-monotonic: For small levels of uncertainty growing uncertainty is reducing the optimal timing threshold while for high levels of uncertainty growing uncertainty is increasing the optimal timing threshold.*

Moreover, one central result of Gryglewicz et al. (2008) is that the negative effect of uncertainty on investment threshold is only observed for investment projects with a finite lifetime. For an infinitely long project an increase of uncertainty will always lead to a postponement of the investment, i.e. a pure positive effect of uncertainty on the investment threshold exist. From equation (13) it is obvious that this is not the case in a sequential bargaining game. Here, $\partial x^*/\partial \sigma$ is not always greater zero for all levels of uncertainty. For $T \to \infty$ we have that:

$$\frac{\partial x^*}{\partial \sigma} = \frac{\sigma\beta}{(\sigma^2\beta^2 + 2r)}\left[\frac{1}{2}\sigma^2\beta(\beta - 1) + \alpha\beta + r\right]\left[1 + \frac{\varepsilon}{(\beta - 1)}\right] - \frac{\varepsilon}{(\beta - 1)^2}\frac{\partial \beta}{\partial \sigma}x^* \quad (16)$$

Hence, $\partial x^*/\partial \sigma$ is smaller than zero $\forall \varepsilon > 0$ once $\sigma \to 0$.

Proposition 3b: *In a real sequential bargaining setting,* i.e. $\varepsilon > 0$, *the influence of uncertainty on the investment threshold stays non-monotonic for an infinitely long investment projects.*

One more result is deducible from equation (15). Because of $\left(1 + \frac{\varepsilon}{\beta-1}\right) > 1$ the parties time the investment inefficiently late which leads to the following proposition:

Proposition 4: *The investment in the project happens inefficiently late if it is bargained sequentially by the two parties and if $\varepsilon > 0$. Specifically, the social inefficiency is the more pronounced the higher the fraction of cost the second player has to bear.*

However, three limiting cases exist in which the sequentially bargained deal becomes social efficient: First, if the second player bears no investment costs his optimal threshold equals the social efficient threshold. As we will see in equation (21) the second player does not generate any surplus under that setting. Hence, this setting is equivalent to the non-game-theoretical model in Gryglewicz et al. (2008) which therefore can be seen as a special and limiting case of our game-theoretical model. Second, situations may occur where postponement of the investment would generate no extra value. In particular, a very large discount rate r implies that the individual will place a high weight on the immediate present. As a consequence, immediate investment represents the second limiting case. Likewise, a negative growth rate, i.e. $\alpha \leq 0$ causes $x(t)$ to remain constant or fall over time and thus it is again optimal to invest immediately if $x_0 > \frac{\delta}{(1-e^{-\delta T})} I$ and never invest otherwise.

In the following, we will give an answer to the question how much of wealth is distributed to the parties? In the continuation region, i.e. $x_0 < x^*$ the generated surplus equals:

$$G(V_0) = G\big(V(x_0)\big) = \left(\frac{\delta}{(1-e^{-\delta T})}x^* - I\right)\left(\frac{\frac{\delta x_0}{(1-e^{-\delta T})}}{\frac{\delta x^*}{(1-e^{-\delta T})}}\right)^{\beta} \tag{17}$$

$$= \left(\frac{\beta}{\beta-1} + \varepsilon\frac{\beta}{(\beta-1)^2} - 1\right)I\left(\frac{x_0}{\left(1+\frac{\varepsilon}{\beta-1}\right)\frac{\beta}{\beta-1}I}\right)^{\beta},$$

Because A is the offering party and thus holds the bargaining power, his expected

profit equals $s_A G(V(x_0))$ with a share of the surplus of:

$$s_A = \frac{\left(\left(1-\frac{(\beta-1)\varepsilon}{(\beta-1)+\varepsilon}\right)\frac{\delta}{(1-e^{-\delta T})}x^* - (1-\varepsilon)I\right)\left(\frac{x_0}{x^*}\right)^{\beta}}{\left(\frac{\delta}{(1-e^{-\delta T})}x^* - I\right)\left(\frac{x_0}{x^*}\right)^{\beta}} = \frac{\varepsilon+\beta-1}{\varepsilon\beta+\beta-1}. \tag{18}$$

In contrast, the reacting party receives the fraction $s_B G(V(x_0))$ with

$$s_B = 1 - s_A = \frac{\varepsilon\beta - \varepsilon}{\varepsilon\beta + \beta - 1}. \tag{19}$$

Because of $s_B = \frac{\varepsilon\beta-\varepsilon}{\varepsilon\beta+\beta-1} = \frac{(\beta-1)\varepsilon}{\varepsilon\beta+\beta-1} < \frac{(\beta-1)+\varepsilon}{\varepsilon\beta+\beta-1} = s_A$ we get the following

proposition:

Proposition 5: *The expected profit for being the offering party is greater than for*

being the reacting party, i.e. $s_B < s_A$.

As is shown in Wong (2007) we have $\frac{\partial\beta}{\partial\sigma} > 0$ for small values of uncertainty and

$\frac{\partial\beta}{\partial\sigma} < 0$ for higher values of uncertainty. With $\frac{\partial s_A}{\partial\beta} = -\frac{\varepsilon^2}{(\beta\varepsilon+\beta-1)^2} < 0$ we can

therefore state the following proposition:

Proposition 6: *Uncertainty has an ambivalent influence on this first-mover-*

advantage. For small levels of uncertainty growing uncertainty is weakening the

first-mover-advantage while for high levels of uncertainty growing uncertainty is

strengthening the first-mover-advantage.

This ambiguous effect is based on two opposing effects, namely the impact of risk-adjusted discounting and the impact of managerial flexibility. For small levels of uncertainty, the discounting effect dominates. Here, an increase in the adjusted return on the investment weakens player A's first-mover advantage because a higher discount rate lowers the future value of the gain to be shared. Consequently, player B profits from any increase in the risk-adjusted rate and due to the linear relationship, i.e. $\mu = r + \lambda \rho \sigma$, an increase in uncertainty raises μ and s_B, respectively.

For significant high levels of uncertainty, however, the impact of managerial flexibility dominates. Obviously, the total surplus generated by the investment project is ceteris paribus the lower the more the investment is delayed. Therefore, controlling the exercise of the real option can be regarded as having some bargaining power. But increasing uncertainty is diminishing this bargaining power of the second mover because an increase in uncertainty increases the total surplus of the investment project. Hence, a postponement of the investment is less threatening to the first-mover, i.e. A, because he also profits from a delayed investment due to a better information set and a higher flexibility value, respectively. As a consequence the first-mover advantage as proxied by the size of s_A increases.

3. Numerical Analysis

In the following our model is analyzed numerically. To be able to compare the results with the existing literature we assume the same values: $r = 0.1, \lambda = 0.4, \rho = 0.7, I = 1, \alpha = 0.08$ and $t_0 = 0$.
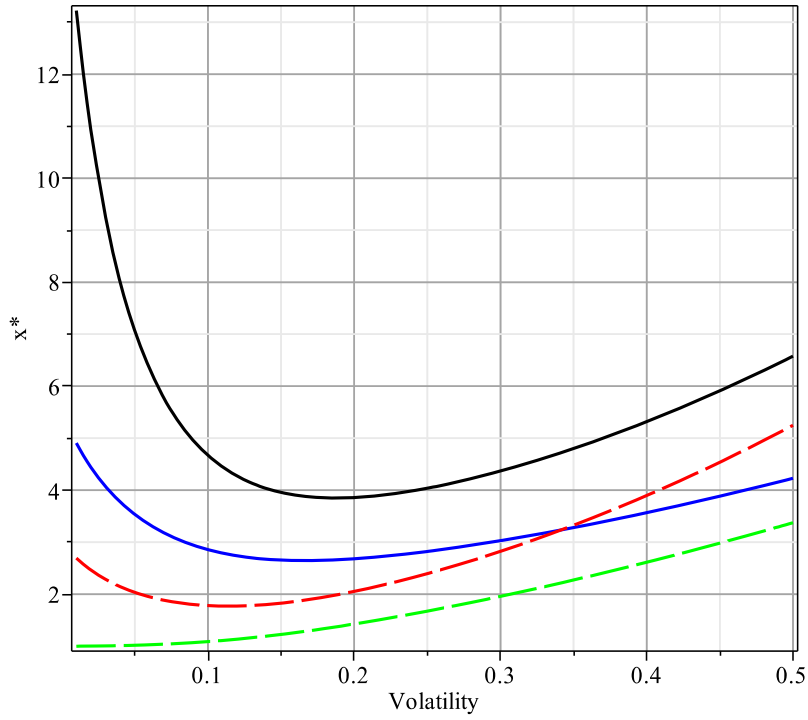

Figure 1: Impact of uncertainty σ on investment threshold x^* for the sequential-bargaining model (solid black curve, ε=0.5, T=10) and the model of Gryglewicz et al. (2008) (solid blue curve, ε=0, T=10). For the limiting case, i.e. $T \rightarrow \infty$, the uncertainty-investment relationship for the sequential bargaining game is expressed by the longdashed red curve. The green longdashed curve replicates the results of Gryglewicz et al. (2008), Wong (2007) and Sarkar (2000) (ε=0).

As Figure (1) depicts the investment will in expectation occur later after sequential bargaining between two parties (black curve) than in the one party case (grey curve), described by Gryglewicz et al. (2008). However, the influence of uncertainty on the expected investment time still shows a U-shape after sequential bargaining. Specifically, an increasing fraction of the investment cost beard by the

second player is shifting the curve upwards and to the right and therefore is resulting in a further postponement of the investment. As already has been stated, the model of Wong (2007) and Sarkar (2000) can be seen as a special case of our sequential-bargained model, indeed, if $\varepsilon = 0$ both curves will be identical. According to Proposition (6) the influence of uncertainty on the shares of the surplus of the two players is ambiguous.
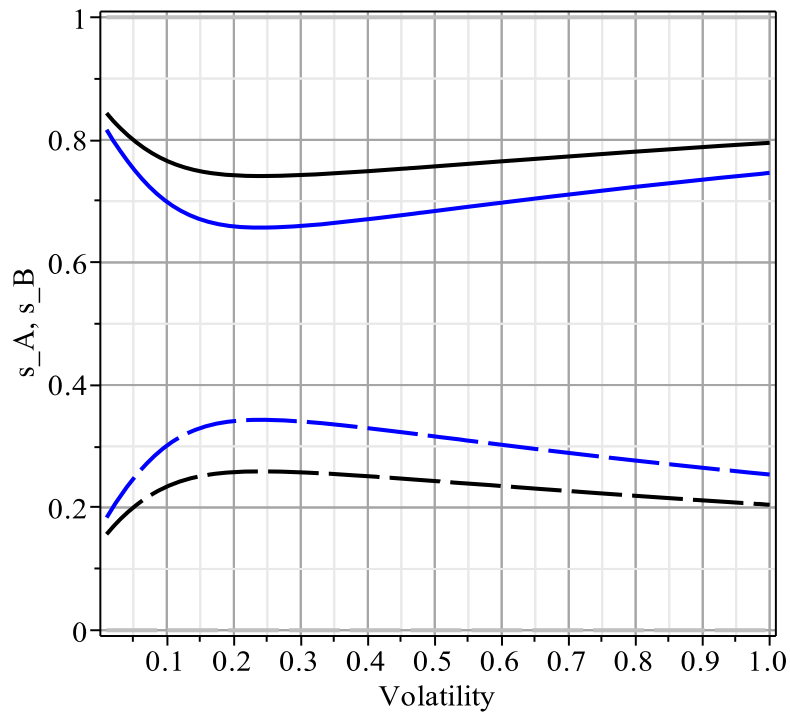

Figure 2: The shares of the surplus of the offering party (s_A, solid) and of the reacting party (s_B, long-dash) depending on the amount of uncertainty, σ, and the size of incurred costs the parties face, $\varepsilon = 0.95$ (blue), $\varepsilon = 0.5$ (black), and $\varepsilon = 0$ (grey).

As can be seen in Figure (2) this influence is following a U-shape, too. Starting with

$$\lim_{\sigma \to 0} s_A = \frac{\frac{r}{\alpha} - 1 + \varepsilon}{\frac{r}{\alpha}\varepsilon + \frac{r}{\alpha} - 1} > \frac{1}{2} \tag{20}$$

the share of the first player is first decreasing with increasing uncertainty. For higher levels of uncertainty, however, the share of the first-player is increasing with increasing uncertainty. Under infinite uncertainty the first-player would get the whole surplus, $\lim_{\sigma \to \infty} s_A = 1$.

From Figure 2 it is also apparent that the ambiguity is controlled by the distribution of the investment costs, i.e. $(1 - \varepsilon)I$ and εI, respectively. In particular, should the first player bear no or moderate costs, i.e. , $0 \leq \varepsilon < 0.5$, the ambiguity is well pronounced. This is due to the fact, that the second player makes his timing decision contingent not only on the size of uncertainty but on the degree of irreversibility as represented by the size of sunk costs, too. Consequently, the first mover profits from B's managerial flexibility the less cost he incurs. In contrast, the higher the fraction of overall costs devoted to A becomes the weaker becomes the ambiguous effect of uncertainty on the shares of the surplus. Hence, should he incur the full cost, the sharing of surplus does no longer depend on the size of uncertainty. In this limiting case, the result resembles the one of a simple ultimatum game.

4. Conclusion

While the standard real option optimization framework and the deduced uncertainty-investment relationship, respectively, are characterized by a game against nature a proper treatment of games between individuals is missing. As an approach to fill this gap we have set up a model that builds on the assumptions of Gryglewicz et al. (2008), Wong (2007) and Sarkar (2000) but treats investment timing as an outcome of a sequential game with two individuals. The results show that a U-shaped investment-uncertainty relationship generally sustains. However, timing of an investment occurs inefficiently late. This inefficiency is increasing with the fraction of investment cost the second player has to bear. Furthermore the results show that in the sequential bargaining game always a first-mover-advantage prevails, i.e. the first player gets a higher fraction of the combined surplus than the second player. The amount of this first-mover-advantage is influenced by uncertainty in a U-shaped pattern, too. While for small values of uncertainty the first-mover-advantage is decreasing with increasing uncertainty for higher values of uncertainty it is increasing with increasing uncertainty.

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